On the strong approximation of non-overlapping m-spacings processes

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Abstract.

In this paper we establish strong approximations of the uniform non-overlapping m-spacings process extending the results of (1). Our methods rely on the (9) invariance principle.

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1 Introduction and Main Result

Let $U_1, U_2, \ldots$, be independent and identically distributed $(i.i.d.)$ uniform $[0,1]$ random variables $(r.v's)$ defined on the same probability space (Ω, A, P). Denote by $0 =: U_{0,n} \leq U_{1,n} \leq \cdots \leq U_{n-1,n} \leq U_{n,n} := 1$, the order statistics of $U_1, U_2, \ldots, U_{n-1}$, and $0, 1$.

The corresponding non-overlapping m-spacings are then defined by

$$\begin{aligned} D_{i,n}^{(m)} &:= U_{im,n} - U_{(i-1)m,n}, \quad 1 \leq i \leq N-1, \\ D_{N,n}^{(m)} &:= 1 - U_{(N-1)m,n}, \end{aligned} \tag{1}$$

where $N = \lfloor n/m \rfloor$, with $\lfloor u \rfloor \leq u < \lfloor u \rfloor + 1$ denoting the integer part of u.
When $m = 1$ i.e $N = n$, the m-spacings reduce to the the usual 1-spacings (or simple spacings) defined by $D_{i,n}^{(1)} = U_{i,n} - U_{i-1,n}, \ i = 1, \ldots, n$. Simple spacings have received a great deal of attention in the literature. We refer to (7), (10; 11), (13), (12), (2) and (3).

It is well known (see, e.g., (10)) that, for any $n \geq 1$, the simple spacings $\{D_{i,n}^{(1)} : 1 \leq i \leq n\}$ form an exchangeable set of random variables such that, for each fixed $t \geq 0$, uniformly over $1 \leq i \leq n$,

$$P(nD_{i,n}^{(1)} \leq t) = P(nD_{1,n}^{1} \leq t) = 1 - \left(1 - \frac{t}{n}\right)^{n-1} \to 1 - e^{-t}, t \geq 0, \tag{2}$$

as n tends to infinity. Then the normalized spacings have the exponential one distribution function.

Throughout the sequel, $m \geq 1$ will denote a fixed integer. In applications it is more convenient to use the normalized non-overlapping m-spacings $\{mND_{i,n}^{(m)} : 1 \leq i \leq N\}$. For a fixed $m \geq 1$, as $n \to \infty$, the distribution function of $mND_{i,n}^{(m)}$ (which is independent of the index i with $1 \leq i \leq N-1$) converges to the distribution function $F^{(m)}$, of a standard gamma random variable with expectation m, given by

$$F^{(m)}(t) := \frac{1}{(m-1)!} \int_0^t x^{m-1} e^{-x} dx = \int_0^t f^{(m)}(t) dt \ \text{ for } \ t \geq 0, \tag{3}$$

with

$$f^{(m)}(t) = \frac{t^{m-1} e^{-t}}{(m-1)!} \ \text{ and } \ F^{(m)}(t) = 0 \ \text{ for } \ t < 0. \tag{4}$$

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For each choice of $m \geq 1$, the empirical m-spacings process is defined by

$$\alpha_n(x) = N^{1/2}\left(\hat{F}_n(x) - F^{(m)}(x)\right),\ x > 0, \tag{5}$$

where $\hat{F}_n(\cdot)$ is the empirical distribution function of $\{mND_{i,n}^{(m)} : 1 \leq i \leq N\}$, defined for $n \geq m$, by

$$\hat{F}_n(x) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left\{mND_{i,n}^{(m)} \leq x\right\}},\ x \in \mathbb{R}, \tag{6}$$

with $\mathbb{1}(A)$ denoting the indicator function of the event A.

We will need the following additional notations and definitions. Let

$$M_{1:n}^{(m)} \leq M_{2:n}^{(m)} \leq \cdots \leq M_{N:n}^{(m)}, \tag{7}$$

be the order statistics of $\{D_{i,n}^{(m)} : 1 \leq i \leq N\}$. The quantile m-spacings function is given by

$$\hat{Q}_n(t) := \begin{cases} mNM_{i,n}^{(m)}, & \text{if } \frac{i-1}{N} < t \leq \frac{i}{N},\ \ i = 1, 2, \ldots, N, \\ 0, & \text{if } t = 0. \end{cases}$$

Let

$$Q^{(m)}(t) = \inf\left\{x \geq 0 : F^{(m)}(x) \geq t\right\}, \tag{8}$$

and $f^{(m)}(t) = \frac{d}{dt}F^{(m)}(t)$. The quantile m-spacings process γ_n is then defined by

$$\gamma_n(t) = N^{1/2} f^{(m)}\left(Q^{(m)}(t)\right)\left(Q^{(m)}(t) - \hat{Q}_n(t)\right), 0 \leq t \leq 1. \tag{9}$$

The aim in this paper is to obtain a refinement of the strong approximation results for α_n and γ_n obtained by (1). Their main tool is the well known (KMT) invariance principle introduced in (8) by Komlós, Major and Tusnády. In our approach we shall make use the refinement of the KMT inequality for the Brownian bridge approximation of uniform empirical and quantile processes presented respectively in (9) and (6). This approach is based on the approximation of the m-spacings process on $(0, a)$, with $a \leq 1$.

In order to prove the invariance principle, we use the same method developed in (1), which is based on the following representation of simple spacings given by (10).

Let $E_1, E_2, \ldots$ denote an $i.i.d.$ sequence of exponential $r.v's$ with mean 1 and set $S_n := \sum_{i=1}^{n} E_i$. Then for each $n > 1$, we have the distributional identity

$$\{U_{i,n} - U_{i-1,n} : 1 \leq i \leq n\} \stackrel{d}{=} \left\{\frac{E_i}{S_n} : 1 \leq i \leq n\right\}. \tag{10}$$

Consequently we obtain the following representation of the non-overlapping m-spacings

$$\begin{aligned}\left\{D_{i,n}^{(m)}, 1 \leq i \leq N-1, D_{N,n}^{(m)}\right\} \stackrel{d}{=} \Bigg\{ & \left(\sum_{\ell=i}^{i+m-1} E_\ell\right)/S_n, \\ & i = 1, m+1, \ldots, \left(\left\lfloor\frac{n}{m}\right\rfloor - 1\right)m + 1, \left(\sum_{\ell=m\lfloor\frac{n}{m}\rfloor+1}^{n} E_\ell\right)/S_n\Bigg\}. \end{aligned} \tag{11}$$

$\lfloor x\rfloor \leq x \leq \lfloor x\rfloor + 1$. In particular, if $n = mN$ is an integer multiple of m, then

$$\left\{D_{i,n}^{(m)}, 1 \leq i \leq N\right\} \stackrel{d}{=} \left\{Y_i/T_N, 1 \leq i \leq N\right\}, \tag{12}$$

where

$$Y_i := \sum_{\ell=(i-1)m+1}^{im} E_\ell, i = 1, 2, \ldots, N, \tag{13}$$

is a sequence of independent identically distributed rv's with distribution function $F^{(m)}$ and $T_N = \sum_{i=1}^N Y_i$.

Now, we denote by G_N the empirical distribution function and by K_N the empirical quantile function of the sequence $Y_1, \ldots, Y_N$, respectively, defined by

$$G_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{Y_i \le x\}}, \text{ for all } x \in \mathbb{R}^+, \tag{14}$$

and

$$K_N(t) := \inf\{x : G_N(x) \ge t\}, \text{ for all } t \in [0,1]. \tag{15}$$

Let β_N and κ_N be the corresponding empirical and quantile processes, respectively, defined by

$$\beta_N(x) := \sqrt{N} \left(G_N(x) - F^{(m)}(x) \right), \text{ for all } x \in \mathbb{R}^+, \tag{16}$$

and

$$\kappa_N(t) := \sqrt{N} f^{(m)} \left(Q^{(m)}(t) \right) \left(Q^{(m)}(t) - K_N(t) \right), \text{ for all } t \in [0,1]. \tag{17}$$

By (12) we have the following representation

$$\{\alpha_{mN}(x), 0 \le x < \infty\} \stackrel{d}{=} \left\{ \alpha_N^1(x) = \beta_N \left(x \frac{T_N}{mN} \right) + \mathcal{R}_N(x), 0 \le x < \infty \right\}, \tag{18}$$

where

$$\mathcal{R}_N(x) = N^{1/2} \left(F^{(m)} \left(x \frac{T_N}{mN} \right) - F^{(m)}(x) \right).$$

In fact:

$$\begin{aligned}
&\{\alpha_{mN}(x), 0 \le x < \infty\} \\
&\stackrel{d}{=} \left\{ N^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left\{ \frac{mN}{T_N} Y_i \le x \right\}} - F^{(m)}(x) \right), 0 \le x < \infty \right\} \\
&= \left\{ N^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left\{ Y_i \le \frac{T_N}{mN} x \right\}} - F^{(m)}(x) \right), 0 \le x < \infty \right\}.
\end{aligned}$$

By adding and subtracting $F^{(m)} \left(\frac{T_N}{mN} x \right)$, in the right side, we obtain

$$\begin{aligned}
&\left\{ N^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left\{ Y_i \le \frac{T_N}{mN} x \right\}} - F^{(m)}(x) \right), 0 \le x < \infty \right\} \\
&= \left\{ N^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left\{ Y_i \le \frac{T_N}{mN} x \right\}} - F^{(m)} \left(\frac{T_N}{mN} x \right) \right) + \mathcal{R}_N(x), 0 \le x < \infty \right\} \\
&= \left\{ N^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left\{ \xi_i \le F^{(m)} \left(\frac{T_N}{mN} x \right) \right\}} - F^{(m)} \left(\frac{T_N}{mN} x \right) \right) + \mathcal{R}_N(x), x \in \mathbb{R}_+ \right\} \\
&= \left\{ \alpha_N \left(F^{(m)} \left(\frac{T_N}{mN} x \right) \right) + N^{1/2} \left(F^{(m)} \left(\frac{T_N}{mN} x \right) - F^{(m)}(x) \right), x \in \mathbb{R}_+ \right\} \\
&= \left\{ \alpha_N^1(x) = \beta_N \left(\frac{T_N}{mN} x \right) + \mathcal{R}_N(x), 0 \le x < \infty \right\}.
\end{aligned}$$

In the same way, by (12), and definition of the empirical quantile function K_N, we have the following representation for γ_{mN}.

$$\{\gamma_{mN}(t), 0 \le t < 1\} \stackrel{d}{=} \left\{ \gamma_N^1(t) = \frac{mN}{T_N} \left(\kappa_N(t) + N^{1/2} \left(\frac{T_N}{mN} - 1 \right) \phi_m(t) \right), 0 \le t < 1 \right\}, \tag{19}$$

and

$$\phi_m(t) = f^{(m)}(Q^{(m)}(t)) Q^{(m)}(t).$$

In fact:

$$\begin{aligned}
&\{\gamma_{mN}(t), 0 \le t < 1\} \\
&\overset{d}{=} \quad \left\{ N^{1/2} f^{(m)}\left(Q^{(m)}(t)\right)\left(Q^{(m)}(t) - \frac{mN}{T_N} Y_{i,N}\right), 0 \le t < 1 \right\} \\
&= \quad \left\{ N^{1/2} f^{(m)}\left(Q^{(m)}(t)\right)\left(Q^{(m)}(t) - \frac{mN}{T_N} K_N(t)\right), 0 \le t < 1 \right\}.
\end{aligned}$$

By added and subtracted $\frac{mN}{T_N} Q^{(m)}(t)$, in the right side, we obtain

$$\begin{aligned}
&\{\gamma_N(t), 0 \le t < 1\} \\
&= \left\{ \frac{mN}{T_N}\kappa_N(t) + N^{1/2} f^{(m)}\left(Q^{(m)}(t)\right)\left(Q^{(m)}(t) - \frac{mN}{T_N} Q^{(m)}(t)\right), 0 \le t < 1 \right\} \\
&= \left\{ \frac{mN}{T_N}\left(\kappa_N(t) + N^{1/2} f^{(m)}\left(Q^{(m)}(t)\right)\left(\frac{T_N}{mN} Q^{(m)}(t) - Q^{(m)}(t)\right)\right), 0 \le t < 1 \right\} \\
&= \left\{ \frac{mN}{T_N}\left(\kappa_N(t) + N^{1/2} f^{(m)}\left(Q^{(m)}(t)\right) Q^{(m)}(t)\left(\frac{T_N}{mN} - 1\right)\right), 0 \le t < 1 \right\} \\
&= \left\{ \frac{mN}{T_N}\left(\kappa_N(t) + N^{1/2}\left(\frac{T_N}{mN} - 1\right) f^{(m)}\left(Q^{(m)}(t)\right) Q^{(m)}(t)\right), 0 \le t < 1 \right\}.
\end{aligned}$$

2 Preliminaries

In the sequel, we will assume, without loss of generality, that the original probability space, on which are defined $U_1, U_2, \ldots$, a sequence of independent uniform $(0,1)$ random variables and $B_1, B_2, \ldots$ a sequence of Brownian bridges. This important assumption is used to prove invariance principles.

Throughout the paper we denote by $\mathcal{A}, \mathcal{B}, A_i, B_i, i = 1, 2, \ldots$ which are appropriate positive constants, and by $\log$ the function $u \mapsto \log_+(u) = \log(u \vee e), \forall\ u \in \mathbb{R}$. Let us recall the following theorem.

Theorem 2.1 *((9)). There exists a sequence of empirical processes β_N based on $Y_1, \ldots, Y_N$ and a sequence of Brownian bridges $\{B_N^{(1)}(t), 0 \le t \le 1\}$ such that, for all $\varepsilon > 0$ and $0 \le a \le 1$, we have*

$$P\left(\sup_{0 \le x \le Q^{(m)}(a)} |\beta_N(x) - B_N^{(1)}(F^{(m)}(x))| \ge \mathcal{A} N^{-1/2}(\log aN)\right) \le \mathcal{B} N^{-\varepsilon}, \tag{20}$$

where $\mathcal{A}$ and $\mathcal{B}$ are positive constants depending on ε and a.

A similar result is needed for the quantile process κ_n. For this, we consider deviations between the quantile process κ_N and the approximating Brownian bridges $\{B_N^{(1)}(t), 0 \le t \le 1\}$ on $[0, a]$, instead of $[0, 1]$. We formulate this idea in the following theorem.

Theorem 2.2 *Let $\{B_N^{(1)}(t), 0 \le t \le 1\}$ be as in of Theorem 2.1. Then for all $\varepsilon > 0$ and $n \ge m$, we have*

$$P\left(\sup_{0 \le t \le a} |\kappa_N(t) - B_N^{(1)}(t)| \ge A_1 N^{-1/4}(\log aN)^{3/4}\right) \le B_1 N^{-\varepsilon}, \tag{21}$$

for all $0 \le a \le 1$, where A_1 and B_1 are positive constants.

We give now, some technical Lemma which we will use to prove our results bellow.

Theorem 2.3 (The Borel-Cantelli lemma) *For any sequence $\{A_n : n \ge 1\} \subseteq A$ of measurable events, we have*

$$\sum_{i=1}^{n} P(A_n) < \infty \quad \Rightarrow \quad P(A_n \quad \text{i.o.}) = 0 \quad \Leftrightarrow \quad P(A_n \quad \text{f.o.}) = 1. \tag{22}$$

$$\sum_{i=1}^{n} P(A_n) = \infty \quad \Rightarrow \quad P(A_n \quad \text{i.o.}) = 1 \quad \Leftrightarrow \quad P(A_n \quad \text{f.o.}) = 0. \tag{23}$$

Where i.o. *and* f.o. *designed respectively, infinitely often and finitely often.*

Lemma 2.4 (lemma 1.2.1 (4)) *For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that the inequality*

$$P\left(\sup_{0\le s\le T-h}\sup_{0\le t\le h}|W(s+t)-W(s)| \ge v\sqrt{h}\right) \le \frac{CT}{h}e^{-\frac{v^2}{2+\varepsilon}}, \tag{24}$$

holds for every positive v, T and $0 < h < T$.

Lemma 2.5 (lemma 1.4.1 (4)) *Let $\{W(t); 0 \le t \le 1\}$ be a Wiener process. Then*

$$B(t) = W(t) - tW(1) \ \ (0 \le t \le 1), \tag{25}$$

is a Brownian bridge.

Lemma 2.6 (lemma 4.4.4 (4)) *Let $\mu(\cdot)$ be a probability measure defined on the Borel sets of the Banach space $D(0,1)\times D(0,1)$, and let ξ (res. η) be $D(0,1)$ valued $r.v$ defined on (Ω_1, A_1, P_1) (res. (Ω_2, A_2, P_2)) with*

$$P_1\{\xi \in A\} = \mu(A \times D(0,1)) \ \ \textit{res.} \ \ P_2\{\eta \in A\} = \mu(D(0,1)\times A), \tag{26}$$

for any Borel set A of $D(0,1)$. There exists a probability measure P defined on $(\Omega_1\times\Omega_2, A_1\times A_2)$ such that

$$P\{(\omega_1,\omega_2)\in\Omega_1\times\Omega_2 : (\xi(\omega_1),\eta(\omega_2))\in B\} = \mu(B), \tag{27}$$

for any Borel set B of $D(0,1)\times D(0,1)$.

3 Local Strong Approximation

We state now our main theorems.

Theorem 3.1 *There exist a sequence $\{W_{mN}, 0 \le t \le 1\}_{N>1}$ of Gaussian processes, such that*

$$EW_{mN}(t) = 0,$$

$$EW_{mN}(t)W_{mN}(s) = \min(t,s) - ts - \frac{1}{m}\phi_m(t)\phi_m(s),$$

and

$$\phi_m(t) = f^{(m)}\left(Q^{(m)}(t)\right)Q^{(m)}(t).$$

Moreover, for each $\varepsilon > 0$, there exists constants $A_2 > 0$ and $B_2 > 0$, such that, for all $n \ge m$ and $a \in [0,1]$ we have

$$P\left(\sup_{0\le t\le a}|\gamma_{mN}(t) - W_{mN}(t)| > A_2 N^{-1/4}(\log aN)^{3/4}\right) \le B_2 N^{-\varepsilon}.$$

Theorem 3.2 *There exist a sequence of Gaussian processes $\{V_n(x), 0 \le x \le \infty\}$, such that*

$$EV_n(x) = 0, \tag{28}$$

and

$$EV_n(x)V_n(y) = \min\left(F^{(m)}(x), F^{(m)}(y)\right) - F^{(m)}(x)F^{(m)}(y) - \frac{1}{m}xyf^{(m)}(x)f^{(m)}(y). \tag{29}$$

Moreover, for all $\varepsilon > 0$ and $a \in [0,1]$ we have

$$P\left(\sup_{0\le x\le Q^{(m)}(a)}|\alpha_n(x) - V_n(x)| \ge A_3 N^{-1/2}(\log aN)\right) \le B_3 N^{-\varepsilon},$$

where $A_3 > 0$ and $B_3 > 0$ are positive constants.

Remark 1 By Borel-Cantelli Lemma and Theorem 2.2 we have

$$\sup_{0\le t\le a}|\gamma_{mN}(t)-W_{mN}(t)|\overset{a.s.}{=}O\left(N^{-1/4}(\log aN)^{3/4}\right). \tag{30}$$

Applying Borel-Cantelli Lemma and Theorem 3.2 we have

$$\sup_{0\le x\le Q^{(m)}(a)}|\alpha_n(x)-V_n(x)|\overset{a.s.}{=}O\left(N^{-1/4}(\log aN)^{3/4}\right). \tag{31}$$

For $a=1$, our results reduce to the results of (1).

4 Proof

4.1 Proof of Theorem 2.2.

Consider the sequence $\xi_i = F^{(m)}(Y_i), i = 1,2,\dots,$ of i.i.d. $U[0,1]$ $r.v's$ and construct the corresponding uniform quantile process defined by

$$U_N(t)=N^{1/2}(t-F^{(m)}(K_N(t))), \tag{32}$$

where Y_i and $K_N(t)$ are defined by (13) and (15) successively. A simple application of theorem (1.1) of (6) with $a=d/n$ and $x=\varepsilon\lambda^{-1}\log aN$, we can find a sequence of Brownian bridges $\{B_N^{(2)}(t), 0\le t\le 1\}$, such that for all $\varepsilon>0$ we have

$$P\left(\sup_{0\le t\le a}|U_N(t)-B_N^{(2)}(t)|\ge A_4N^{-1/2}(\log aN)\right)\le B_4N^{-\varepsilon}, \tag{33}$$

where A_4, B_4 are positive constants depending on ε and a. Furthermore, we have for all $0\le a\le 1$,

$$P\left(\sup_{0\le t\le a}|B_N^{(2)}(t)|>x\right)\le 2e^{-2x^2},\ \ x\ge 0. \tag{34}$$

The last inequality together with (33) implies that

$$P\left(\sup_{0\le t\le a}|U_N(t)|\ge\left(\frac{1}{2}\varepsilon(\log aN)\right)^{1/2}+A_4N^{-1/2}(\log aN)\right)\le(2+B_4)\,N^{-\varepsilon}. \tag{35}$$

We will prove in the next lemma that $U_N(t)$, as defined in (32), can be approximated by $B_N^{(1)}$ as well.

Lemma 4.1 *For all $\varepsilon>0$ we have*

$$P\left(\sup_{0\le t\le a}|U_N(t)-B_N^{(1)}(t)|\ge A_5N^{-1/2}(\log aN)^{3/4}\right)\le B_5N^{-\varepsilon}, \tag{36}$$

where A_5 and B_5 are positive constants.

Proof of Lemma 4.1. Let $\xi_{1,N},\dots,\xi_{N,N}$ denote the order statistics of $\xi_1,\dots,\xi_N$. By Theorem 2.1 and the fact that $\beta_N(Q^{(m)}(\xi_{i,N}))=U_N(\frac{i}{N})$, we have, for each $0<a\le 1$

$$P\left\{\max_{0\le i\le aN}\left|U_N\left(\frac{i}{N}\right)-B_N^{(1)}\left(\xi_{i,N}\right)\right|>\mathcal{A}N^{-1/2}\left(\log aN\right)\right\}\le\mathcal{B}N^{-\varepsilon}. \tag{37}$$

On the other hand, from (35) we have

$$P\left\{\max_{0\le i\le aN}\left|\frac{i}{N}-\xi_{i,N}\right|\ge N^{-1/2}\left(\frac{\varepsilon}{2}(\log aN)\right)^{1/2}+A_4N^{-1}\left(\log aN\right)\right\}\le(2+B_4)N^{-\varepsilon}. \tag{38}$$

Now, Lemma $1.2.1$ and Lemma $1.4.1$ of (4) allow us to write

$$P\left\{\sup_{0\le i\le N-N^{1/2}(\log aN)}\ \sup_{0\le s\le N^{-1/2}(\log aN)}\left|B_N^{(1)}\left(\frac{i}{N}+s\right)-B_N^{(1)}\left(\frac{i}{N}\right)\right|\right.$$
$$\left.>A_6N^{-1/4}(\log aN)^{3/4}\right\}\le B_6N^{-\varepsilon},$$

This, combined with (38), implies that

$$P\left\{\max_{0\le i\le aN}|B_N^{(1)}\left(\frac{i}{N}\right)-B_N^{(1)}\left(\xi_{i,N}\right)|>A_7N^{-1/4}(\log aN)^{3/4}\right\}\le B_7N^{-\varepsilon}. \tag{39}$$

Lemma 4.1 follows from the fact that

$$\left|U_N(t)-U_N\left(\frac{i}{N}\right)\right|\le N^{-1/2}\ \text{ for }\ \frac{i-1}{N}<t<\frac{i}{N}. \tag{40}$$

■

We return now to the proof of Theorem 2.2. Following (1), we have

$$\sup_{0<t<\infty}F^{(m)}(t)(1-F^{(m)}(t))\frac{|(f^{(m)})'(t)|}{(f^{(m)})^2(t)}\quad\le\quad\gamma, \tag{41}$$

together with

$$\lim_{t\to\infty}F^{(m)}(t)(1-F^{(m)}(t))\frac{|(f^{(m)})'(t)|}{(f^{(m)})^2(t)}=1, \tag{42}$$

$$\lim_{t\to 0}F^{(m)}(t)(1-F^{(m)}(t))\frac{|(f^{(m)})'(t)|}{(f^{(m)})^2(t)}=1, \tag{43}$$

for some $\gamma=\gamma(m)<\infty$.

By the mean value theorem, we obtain

$$\kappa_N(t)-U_N(t)=U_N(t)\left(\frac{f^{(m)}(Q^{(m)}(t))}{f^{(m)}(Q^{(m)}(\theta_{t,N}))}-1\right), \tag{44}$$

for some $\theta_{t,N}$ such that $|\theta_{t,N}-t|<N^{-1/2}|U_N(t)|$. In Theorem 1.5.1 in (5), it is proved that

$$\begin{aligned}&P\left(\sup_{c\le t\le 1-c}\left|\frac{f^{(m)}(Q^{(m)}(t))}{f^{(m)}(Q^{(m)}(\theta_{t,N}))}-1\right|>\delta\right)\\ &\le\quad 4([\gamma]+1)\{\exp(-Nch((1+\delta)^{1/2([\gamma]+1)}))\\ &\qquad+\exp(-Nch((1+\delta)^{-1/2([\gamma]+1)}))\},\end{aligned} \tag{45}$$

for all $\delta>0, 0<c<1$ and $N\ge 1$, where $h(x)=x+\log(1/x)-1, x>0$.
Moreover, there exist a $\delta_0>0$ such that

$$h((1+\delta)^{\mp 1/2([\gamma]+1)})\ge\frac{1}{8}([\gamma]+1)^2\delta^2,\ \ 0<\delta<\delta_0. \tag{46}$$

Let $\delta_N:=(8\varepsilon)^{1/2}([\gamma]+1)^{-1}N^{-1/4}(\log aN)^{1/2}$, and $C^{(1)}:=C_N^{(1)}:=N^{-1/2}$.
By the above inequality and (45) we obtain that, for N sufficiently large, that

$$P\left(\sup_{C_N^{(1)}\le t\le 1-C_N^{(1)}}\left|\frac{f^{(m)}(Q^{(m)}(t))}{f^{(m)}(Q^{(m)}(\theta_{t,N}))}-1\right|>\delta_N\right)\le 8([\gamma]+1)N^{-\varepsilon}. \tag{47}$$

Combining (44), (35) and (47), we obtain that, for N sufficiently large

$$P\left(\sup_{C_N^{(1)}\le t\le a-C_N^{(1)}}|\kappa_N(t)-U_N(t)|>A_8N^{-1/4}(\log aN)^{3/4}\right)\le B_8N^{-\varepsilon}. \tag{48}$$

To complete the proof of Theorem 2.2, we replace $\log N$ in the proof of the Theorem B of (1) by $(\log aN)$. ■

To prove Theorems 3.1 and 3.2, we will make use of Lemma 4.2 and 4.3 bellow.

Lemma 4.2 *We have, for each $\varepsilon > 0$, and all $n \geq m$ sufficiently large*

$$P\left(\left|N^{1/2}\left(\frac{T_N}{mN}-1\right)-\frac{1}{m}\int_0^\infty tdB_N^{(1)}\left(F^{(m)}(t)\right)\right| > A_9 N^{-1/2}\left(\log aN\right)^2\right) \leq B_9 N^{-\varepsilon}. \tag{49}$$

where $A_9 = A_9(\varepsilon) = 4(1/2+\varepsilon)\mathcal{A}$ and $B_9 = 8\sqrt{2}+\mathcal{B}$ denote positive constants.

Proof of Lemma 4.2.

We have,

$$\frac{T_N}{mN} = \frac{1}{mN}\sum_{i=1}^{N} Y_i = \frac{1}{m}\int_0^\infty tdG_N(t) \text{ and } \int_0^\infty tdF^{(m)}(t) = m. \tag{50}$$

Hence

$$N^{1/2}\left(\frac{T_N}{N}-m\right) = \int_0^\infty td\beta_N(t) = -\int_0^\infty \beta_N(t)dt. \tag{51}$$

Let λ_N be a sequence of positive numbers and consider the following decomposition

$$\begin{aligned}\left|\int_0^\infty \beta_N(t)dt - \int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right| \quad &\leq \quad \int_0^{\lambda_N}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|dt \\ &\quad + \int_{\lambda_N}^\infty |B_N^{(1)}(F^{(m)}(t))|dt \\ &\quad + \int_{\lambda_N}^\infty |\beta_N(t)|dt.\end{aligned}$$

We know that

$$E\left(\beta_N(t)\right) = E\left(B_N^{(1)}(F^{(m)}(t))\right) = 0, \tag{52}$$

$$Var\left(\beta_N(t)\right) = E\left[\left(\beta_N(t)\right)^2\right] = F^{(m)}(t)(1-F^{(m)}(t)), \tag{53}$$

and

$$Var\left(B_N^{(1)}(F^{(m)}(t)\right) = E\left[\left(B_N^{(1)}(F^{(m)}(t)\right)^2\right] = F^{(m)}(t)(1-F^{(m)}(t)). \tag{54}$$

By Fubini theorem's and Cauchy-Schwartz inequality we obtain

$$\begin{aligned}E\int_{\lambda_N}^\infty |\beta_N(t)|dt \quad &= \quad \int_{\lambda_N}^\infty E|\beta_N(t)|dt \\ &\leq \quad \int_{\lambda_N}^\infty (F^{(m)}(t)(1-F^{(m)}(t)))^{1/2}dt,\end{aligned} \tag{55}$$

and

$$\begin{aligned}E\int_{\lambda_N}^\infty |B_N^{(1)}(F^{(m)}(t))|dt \quad &= \quad \int_{\lambda_N}^\infty E|B_N^{(1)}(F^{(m)}(t))|dt \\ &\leq \quad \int_{\lambda_N}^\infty (F^{(m)}(t)(1-F^{(m)}(t)))^{1/2}dt.\end{aligned} \tag{56}$$

By (1), there exists $t_0 > 0$ such that

$$1-F^{(m)}(t) \leq 2\exp\left(-\frac{t}{2}\right), \text{ if } t \geq t_0. \tag{57}$$

Hence, provided that $\lambda_N \geq t_0$, by (57) and the fact that

$$F^{(m)}(t) \leq 1 \text{ for all } t > 0, \tag{58}$$

the left hand sides of (55) and (56) are bounded above by $4\sqrt{2}\exp(-\lambda_N/4)$.

Indeed,

$$\begin{aligned} E\left(\left|B_N^{(1)}(F^{(m)}(t))\right|\right) &\leq (F^{(m)}(t)(1-F^{(m)}(t)))^{1/2} \\ &\leq \sqrt{2}\exp(-t/4), \end{aligned}$$

and by using (56), we have

$$\begin{aligned} E\left(\int_{\lambda_N}^{\infty}|B_N^{(1)}(F^{(m)}(t))|dt\right) &\leq \sqrt{2}\int_{\lambda_N}^{\infty}\exp(-t/4)dt \\ &= 4\sqrt{2}\exp(-\lambda_N/4). \end{aligned}$$

In the same way

$$E\left(\int_{\lambda_N}^{\infty}|\beta_N(t)|dt\right) \leq 4\sqrt{2}\exp(-\lambda_N/4).$$

By choosing $\lambda_N = 4(\frac{1}{2}+\varepsilon)(\log aN)$, Markov inequality gives

$$P\left(\int_{4(\frac{1}{2}+\varepsilon)(\log aN)}^{\infty}|\beta_N(t)|dt > a^{-(1/2+\varepsilon)}N^{-1/2}\right) \leq 4\sqrt{2}N^{-\varepsilon}, \tag{59}$$

and

$$P\left(\int_{4(\frac{1}{2}+\varepsilon)(\log aN)}^{\infty}|B_N^{(1)}(F^{(m)}(t))|dt > a^{-(1/2+\varepsilon)}N^{-1/2}\right) \leq 4\sqrt{2}N^{-\varepsilon}. \tag{60}$$

By Theorem (2.1) we can prove that

$$P\left(\int_0^{\lambda_N}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|dt > \lambda_N\mathcal{A}N^{-1/2}(\log aN)\right) \leq \mathcal{B}N^{-\varepsilon}. \tag{61}$$

In fact:

$$\begin{aligned} \int_0^{\lambda_N}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|dt &\leq \sup_{0\leq x\leq Q^{(m)}(a)}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|\int_0^{\lambda_N}dt \\ &= \lambda_N\sup_{0\leq x\leq Q^{(m)}(a)}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|. \end{aligned}$$

By the theorem (2.1), we have

$$\begin{aligned} & P\left(\int_0^{\lambda_N}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|dt > \lambda_N\mathcal{A}N^{-1/2}(\log aN)\right) \\ \leq\ & P\left(\lambda_N\sup_{0\leq x\leq Q^{(m)}(a)}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|dt > \lambda_N\mathcal{A}N^{-1/2}(\log aN)\right) \\ =\ & P\left(\sup_{0\leq x\leq Q^{(m)}(a)}\left|\beta_N(t)-B_N^{(1)}(F^{(m)}(t))\right|dt > \mathcal{A}N^{-1/2}(\log aN)\right) \\ \leq\ & \mathcal{B}N^{-\varepsilon}. \end{aligned}$$

Let $\Lambda_1 = 2a^{-(1/2+\varepsilon)}N^{-1/2}$ and $\Lambda_2 = \lambda_N\mathcal{A}N^{-1/2}(\log aN) = 4(1/2+\varepsilon)\mathcal{A}N^{-1/2}(\log aN)^2$. Then

$$\Lambda_1+\Lambda_2 = 4(1/2+\varepsilon)\mathcal{A}N^{-1/2}(\log aN)^2\,(1+o(1))\,.$$

Lemma 4.2 now follows by combining the above three inequalities (55), (56) and (61).

$$\begin{aligned}
& P\left(\left|\int_0^\infty \left(\beta_N(t) - B_N^{(1)}(F^{(m)}(t))\right)dt\right| > \Lambda_1 + \Lambda_2\right) \\
\leq \;& P\left(\left|\int_0^\lambda \left(\beta_N(t) - B_N^{(1)}(F^{(m)}(t))\right)dt\right| > 4(1/2+\varepsilon)\mathcal{A}N^{-1/2}(\log_+ aN)^2\right) \\
& +P\left(\left|\int_\lambda^\infty \left(B_N^{(1)}(F^{(m)}(t))\right)dt\right| > a^{-(1/2+\varepsilon)}N^{-1/2}\right) \\
& +P\left(\left|\int_\lambda^\infty (\beta_N(t))\,dt\right| > a^{-(1/2+\varepsilon)}N^{-1/2}\right) \\
\leq \;& 4\sqrt{2}N^{-\varepsilon} + 4\sqrt{2}N^{-\varepsilon} + \mathcal{B}.
\end{aligned}$$

If we pose $A_9 = A_9(\varepsilon) = 4(1/2+\varepsilon)\mathcal{A}$ and $B_9 = 8\sqrt{2} + \mathcal{B}$, and the proof of lemma 4.2 is now complete. ■

Lemma 4.3 *For each $\varepsilon > 0$ and $n \geq m$, we have, uniformly over $0 \leq a \leq 1$*

$$\begin{aligned}
P\Bigg(\sup_{0\leq x\leq Q^{(m)}(a)} \left|B_N^{(1)}\left(F^{(m)}\left(x\frac{T_N}{mN}\right)\right) - B_N^{(1)}(F^{(m)}(x))\right| & \\
> A_{10}N^{-1/4}(\log aN)^{3/4}\Bigg) \quad & \leq \quad B_{10}N^{-\varepsilon},
\end{aligned}$$

where A_{10} and B_{10} are positive constants.

Proof of lemma 4.3. The random variable $\int_0^\infty B_N^{(1)}(F^{(m)}(t))\,dt$ has a normal distribution, with expectation 0 and finite variance, given by

$$\sigma_1^2 = E\left\{\left(\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right)^2\right\} < \infty. \tag{62}$$

Hence

$$P\left(\frac{1}{\sigma_1}\left|\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right| > (2\varepsilon\log aN)^{1/2}\right) \leq 2N^{-\varepsilon}. \tag{63}$$

This inequality and Lemma 4.2 imply that

$$P\left(\left|\frac{T_N}{mN} - 1\right| > A_{11}N^{-1/2}(\log aN)^{1/2}\right) \leq B_{11}N^{-\varepsilon}. \tag{64}$$

Where $A_{11} = A_{11}(\varepsilon) = \left(2m^{-2}\sigma_1^2\varepsilon\right)^{1/2}$ and $B_{11} = 2 + B_9$. In fact:

$$A_9N^{-1/2}(\log aN)^{1/2} + (2m^{-2}\sigma_1^2\varepsilon\log aN)^{1/2} = (2m^{-2}\sigma_1^2\varepsilon)^{1/2}(\log aN)^{1/2}(1+o(1)).$$

So the probability (64) is the same as

$$P\left(\left|N^{1/2}\left(\frac{T_N}{mN} - 1\right)\right| > A_9N^{-1/2}(\log aN)^{1/2} + (2m^{-2}\sigma_1^2\varepsilon\log aN)^{1/2}\right).$$

By Lemma 4.2 and inequality (63), it was

$$\begin{aligned}
& P\left(\left|N^{1/2}\left(\frac{T_N}{mN} - 1\right)\right| > A_9N^{-1/2}(\log aN)^{1/2} + (2m^{-2}\sigma_1^2\varepsilon\log aN)^{1/2}\right) \\
\leq \;& P\left(\left|N^{1/2}\left(\frac{T_N}{mN} - 1\right) - \frac{1}{m}\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right| > A_9N^{-1/2}(\log aN)^{1/2}\right) \\
& +P\left(\left|\frac{1}{m}\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right| > (2m^{-2}\sigma_1^2\varepsilon\log aN)^{1/2}\right) \\
= \;& P\left(\left|N^{1/2}\left(\frac{T_N}{mN} - 1\right) - \frac{1}{m}\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right| > A_9N^{-1/2}(\log aN)^{1/2}\right) \\
& +P\left(\left|\frac{1}{\sigma_1}\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right| > (2\varepsilon\log aN)^{1/2}\right) \\
\leq \;& (B_9+2)N^{-\varepsilon}.
\end{aligned}$$

By first order Taylor expansion we have

$$\left|F^{(m)}\left(x\frac{T_N}{mN}\right)-F^{(m)}(x)\right|=xf^{(m)}(x_N)\left|\frac{T_N}{mN}-1\right|, \tag{65}$$

where $|x_N-x|\le\left|\frac{T_N}{mN}-1\right|$. Let $0<\delta<1$ and define $A_N(\delta)$ by

$$A_N(\delta)=\left\{\omega:\left|\frac{T_N}{mN}-1\right|\le\delta\right\}. \tag{66}$$

Now, by choosing N sufficiently large so that $A_{11}N^{-1/2}(\log aN)^{1/2}\le\delta$, and using (64) we get that $P\left(A_N^c(\delta)\right)\le B_{11}N^{-\varepsilon}$. In addition, we have for each $x_N\in A_N(\delta)$,

$$xf^{(m)}(x_N)\le\frac{(1+\delta)^{m-1}}{\Gamma(m)}x^m e^{-(1-\delta)x}, \tag{67}$$

which is bounded on $[0,\infty)$. Now, if

$$A_{12}=A_{11}.\sup_{0\le x\le Q^{(m)}(a)}\frac{(1+\delta)^{m-1}}{\Gamma(m)}x^m e^{-(1-\delta)x}, \tag{68}$$

then

$$\begin{aligned}
&P\left(\sup_{0\le x\le Q^{(m)}(a)}\left|F^{(m)}\left(x\frac{T_N}{mN}\right)-F^{(m)}(x)\right|>A_{12}N^{-1/2}(\log aN)^{1/2}\right)\\
&\le\quad P\left(A_N^c(\delta)\right)\\
&\qquad+P\left(A_N(\delta)\ \text{ and}\left\{\sup_{0\le x\le Q^{(m)}(a)}\left|F^{(m)}\left(x\frac{T_N}{mN}\right)-F^{(m)}(x)\right|>A_{12}N^{-1/2}(\log aN)^{1/2}\right\}\right)\\
&\le\quad P\left(A_N^c(\delta)\right)\\
&\qquad+P\left(A_N(\delta)\ \text{ and}\left\{\sup_{0\le x\le Q^{(m)}(a)}xf^{(m)}(x_N)\left|\frac{T_N}{mN}-1\right|>A_{12}N^{-1/2}(\log aN)^{1/2}\right\}\right)\\
&\le\quad B_{11}N^{-\varepsilon}+P\left(A_N(\delta)\ \text{ and}\left\{\left|\frac{T_N}{mN}-1\right|>A_{11}N^{-1/2}(\log aN)^{1/2}\right\}\right)\\
&\le\quad B_{11}N^{-\varepsilon},\ \text{ for large enough } N.
\end{aligned} \tag{69}$$

Now, (69) combined with Lemma 1.1.1 of (4) implies that

$$\begin{aligned}
&P\left(\sup_{0\le x\le Q^{(m)}(a)}\left|B_N^{(1)}\left(F^{(m)}\left(x\frac{T_N}{mN}\right)\right)-B_N^{(1)}(F^{(m)}(x))\right|>A_{10}\frac{(\log aN)^{3/4}}{N^{1/2}}\right)\\
&=\quad P\left(\sup_{0\le x\le Q^{(m)}(a)}\left|B_N^{(1)}\left(F^{(m)}(x)+F^{(m)}\left(x\frac{T_N}{mN}\right)-F^{(m)}(x)\right)-B_N^{(1)}(F^{(m)}(x))\right|\right.\\
&\qquad\left.>A_{10}N^{-1/2}(\log aN)^{3/4}\right)\\
&\le\quad P\left(\sup_{0\le t\le 1-A_{12}N^{-1/2}(\log aN)^{1/2}}\ \sup_{0\le s\le A_{12}N^{-1/2}(\log aN)^{1/2}}\left|B_N^{(1)}(t+s)-B_N^{(1)}(t)\right|\right.\\
&\qquad\left.>\frac{A_{10}}{\sqrt{A_{12}}}(\log aN)^{1/2}\left(A_{12}N^{-1/2}(\log aN)^{1/2}\right)^{1/2}\right)+B_{11}N^{-\varepsilon}\\
&\le\quad B_{10}N^{-\varepsilon},
\end{aligned} \tag{70}$$

This completes the proof of Lemma 4.3. ■

4.2 Proof of Theorem 3.1.

By the representation (12) we get

$$\left\{\gamma_{mN}(t),0\le t<1\right\}\stackrel{d}{=}\left\{\gamma_N^1(t),0\le t<1\right\}. \tag{71}$$

We want to prove the inequality

$$P\left(\sup_{0\le t\le a}|\gamma_N^1(t)-W_N^*(t)|>A_2N^{-1/4}(\log aN)^{3/4}\right)\le B_2N^{-\varepsilon}, \tag{72}$$

where

$$W_N^*(t):=B_N^{(1)}(t)-\frac{\phi^{(m)}(t)}{m}\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt. \tag{73}$$

First we observe that

$$\begin{aligned}
&\gamma_N^1(t)-\left(B_N^{(1)}(t)-\frac{\phi^{(m)}(t)}{m}\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right)\\
&= \kappa_N(t)-B_N^{(1)}(t)+\left(\left(\frac{T_N}{mN}\right)^{-1}-1\right)\kappa_N(t)\\
&\quad+\phi^{(m)}(t)N^{1/2}\left(\left(\frac{T_N}{mN}\right)-1\right)\left(\left(\frac{T_N}{mN}\right)^{-1}-1\right)\\
&\quad-\frac{\phi^{(m)}(t)}{m}\left(N^{1/2}\left(m-\frac{T_N}{N}\right)-\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right).
\end{aligned} \tag{74}$$

Now, by Theorem 2.2 we have

$$P\left(\sup_{0\le t\le a}|\kappa_N(t)-B_N^{(1)}(t)|\ge A_1N^{-1/4}(\log aN)^{3/4}\right)\le B_1N^{-\varepsilon}. \tag{75}$$

Noting that

$$\sup_{0\le t\le a}\phi^{(m)}(t)=\sup_{0\le x\le Q(a)}xf^{(m)}(x)<\infty. \tag{76}$$

Let $A_{13}=A_9\sup_{0\le x\le Q(a)}xf^{(m)}(x)$, by Lemma 4.2 and (76) we get

$$\begin{aligned}
&P\left(\sup_{0\le t\le a}\left|N^{1/2}\left(1-\frac{T_N}{mN}\right)\phi^{(m)}(t)-\frac{\phi^{(m)}(t)}{m}\int_0^\infty B_N^{(1)}\left(F^{(m)}(t)dt\right)\right|\right.\\
&\qquad\qquad\left.>A_{13}N^{-1/2}\left(\log aN\right)^2\right)\le B_9N^{-\varepsilon}.
\end{aligned} \tag{77}$$

Moreover, we have

$$\left(\left(\frac{T_N}{mN}\right)^{-1}-1\right)\kappa_N(t)=-\left(\left(\frac{T_N}{mN}\right)-1\right)\kappa_N(t)+\left(\left(\frac{T_N}{mN}\right)-1\right)^2\frac{T_N}{mN}\kappa_N(t). \tag{78}$$

First, we have

$$\begin{aligned}
&P\left(\sup_{0\le t\le a}\left|\left(\left(\frac{T_N}{mN}\right)-1\right)\kappa_N(t)\right|>\left(A_{11}N^{-1/2}(\log aN)^{1/2}\right)\right.\\
&\qquad\left.\times\left(\left(\frac{1}{2}\varepsilon(\log aN)\right)^{1/2}+A_1N^{-1/4}(\log aN)^{3/4}\right)\right)\\
&\le P\left(\left|\left(\left(\frac{T_N}{mN}\right)-1\right)\right|>\left(A_{11}N^{-1/2}(\log aN)^{1/2}\right)\right)\\
&\quad+P\left(\sup_{0\le t\le a}|\kappa_N(t)|>\left(\left(\frac{1}{2}\varepsilon(\log aN)\right)^{1/2}+A_1N^{-1/4}(\log aN)^{3/4}\right)\right)\\
&\le P\left(\left|\left(\left(\frac{T_N}{mN}\right)-1\right)\right|>\left(A_{11}N^{-1/2}(\log aN)^{1/2}\right)\right)\\
&\quad+P\left(\sup_{0\le t\le a}\left|\kappa_N(t)-B_N^1(t)\right|>\left(A_1N^{-1/4}(\log aN)^{3/4}\right)\right)\\
&\quad+P\left(\sup_{0\le t\le a}\left|B_N^{(1)}(t)\right|>\left(\frac{1}{2}\varepsilon(\log aN)\right)^{1/2}\right)\\
&\le B_{11}N^{-\varepsilon}+B_1N^{-\varepsilon}+2N^{-\varepsilon}\\
&\le B_{14}N^{-\varepsilon}.
\end{aligned} \tag{79}$$

From the law of large numbers; T_N/N tends to m, as n tends to infinity. Then T_N/Nm tends to one when n tends to infinity. On the other hand, we remark, if $T_N/Nm \geq 1/2$, then $Nm/T_N \leq 2$. We can see that

$$\begin{aligned}
&P\left(\sup_{0\leq t\leq a}\left|\left(\left(\frac{T_N}{mN}\right)-1\right)^2\frac{mN}{T_N}\kappa_N(t)\right| > \left(2A_{11}^2N^{-1}(\log aN)\right)\right.\\
&\qquad\left.\times\left(\left(\frac{1}{2}\varepsilon(\log aN)\right)^{1/2}+A_1N^{-1/4}(\log aN)^{3/4}\right)\right)\\
&\leq\ B_{14}N^{-\varepsilon}.
\end{aligned}\tag{80}$$

Using (79) and (80), we obtain

$$P\left(\sup_{0\leq t\leq a}\left|\left(\left(\frac{T_N}{mN}\right)^{-1}-1\right)\kappa_N(t)\right| > A_{14}N^{-1/4}(\log aN)^{3/4}\right)\leq B_{14}N^{-\varepsilon}.\tag{81}$$

Moreover we have

$$\begin{aligned}
&\phi^{(m)}(t)N^{1/2}\left(\left(\frac{T_N}{mN}\right)-1\right)\left(\left(\frac{T_N}{mN}\right)^{-1}-1\right)\\
&=\ -\phi^{(m)}(t)N^{1/2}\left(\left(\frac{T_N}{mN}\right)-1\right)^2+\phi^{(m)}(t)N^{1/2}\left(\left(\frac{T_N}{mN}\right)-1\right)^3\frac{T_N}{mN}.
\end{aligned}$$

Now, on $A_N(\delta)$, $\sup_{0\leq t\leq a}\phi^{(m)}(t)=\mathcal{M}<\infty$. Taking $A_{15}=A_{11}^2\mathcal{M}$ and applying the technique used in line 2 of (69) we get, by (64), that

$$P\left(\sup_{0\leq t\leq a}\phi^{(m)}(t)N^{1/2}\left|\left(\left(\frac{T_N}{mN}\right)-1\right)^2\right| > A_{15}N^{-1/2}(\log aN)\right)\leq B_{11}N^{-\varepsilon}.\tag{82}$$

Let $A_{16}=2A_{11}^3\mathcal{M}$. Using the same arguments, we see that

$$P\left(\sup_{0\leq t\leq a}\phi^{(m)}(t)N^{1/2}\left|\left(\left(\frac{T_N}{mN}\right)-1\right)^3\right|\frac{mN}{T_N} > A_{16}N^{-1}(\log aN)^{3/2}\right)\leq B_{11}N^{-\varepsilon}.\tag{83}$$

From (82) and (83), we obtain

$$\begin{aligned}
&P\left(\sup_{0\leq t\leq a}\phi^{(m)}(t)N^{1/2}\left(\left(\frac{T_N}{mN}\right)-1\right)\left(\left(\frac{T_N}{mN}\right)^{-1}-1\right)\right.\\
&\qquad\qquad\left.> A_{17}N^{-1/2}(\log aN)\right)\ \leq\ B_{17}N^{-\varepsilon}.
\end{aligned}\tag{84}$$

Now, combining (74), (75), (77), (81) and (84) we get

$$\begin{aligned}
&P\left(\sup_{0\leq t\leq a}\left|\gamma_N^1(t)-\left(B_N^{(1)}(t)-\frac{\phi^{(m)}(t)}{m}\int_0^\infty B_N^{(1)}(F^{(m)}(t))dt\right)\right|\right.\\
&\qquad\qquad\left.> A_2N^{-1/4}(\log aN)^{3/4}\right)\leq B_2N^{-\varepsilon}.
\end{aligned}\tag{85}$$

By Lemma 4.4.4 of (4) and (19), we can define a sequence of Gaussian process $\{W_{mN}(t), 0\leq t\leq 1\}$, $N=1,2,\ldots$ such that for each N, we have

$$\{\gamma_{mN}(t), W_{mN}(s), 0\leq t,s\leq 1\}\stackrel{d}{=}\left\{\gamma_N^1(t), W_N^*(t), 0\leq t,s\leq 1\right\}.\tag{86}$$

This completes the proof of Theorem 3.1. ■

4.3 Proof of Theorem 3.2.

We are going to give the main steps of the proof. The details are the same as in theorem 3.1. Assume first that $n = mN$. Representation (18) for the empirical process of m-spacings, our aim is to prove the following

$$P\left(\sup_{0\le x\le Q^{(m)}(a)} |\alpha_N^1(x) - V_N^*(x)| \ge A_3 N^{-1/2}(\log aN)\right) \le B_3 N^{-\varepsilon}, \tag{87}$$

where

$$V_N^*(x) = B_N^{(1)}(F^{(m)}(x)) - \frac{1}{m} x f^{(m)}(x) \int_0^\infty B_N^{(1)}(F^{(m)}(y))dy. \tag{88}$$

By taking the second order Taylor expansion in the second term of (18), we get

$$\begin{aligned}
\alpha_N^1(x) - V_N^*(x) &= \beta_N\left(x\frac{T_N}{mN}\right) - B_N^{(1)}\left(F^{(m)}\left(x\frac{T_N}{mN}\right)\right) \\
&+ \quad B_N^{(1)}\left(F^{(m)}\left(x\frac{T_N}{mN}\right)\right) - B_N^{(1)}(F^{(m)}(x)) + N^{1/2}\left(\frac{T_N}{mN} - 1\right)^2 x^2 f'^{(m)}(x_N) \\
&+ \quad \frac{xf^{(m)}(x)}{m}\left(N^{1/2}\left(\frac{T_N}{mN} - 1\right) - \int_0^\infty t dB_N^{(1)}(F^{(m)}(t))\right),
\end{aligned}$$

where $|x_N - x| \le x\left|\frac{T_N}{mN} - 1\right|$. Making use of Lemmas 4.2 and 4.3, together with Theorem 2.1 we obtain (87). Hence together with Lemma 4.4.4 of (4), we can define a sequence of Gaussian processes $\{V_{mN}(x), 0 \le x < \infty\}, N = 1, 2, \ldots$, such that for each N we have

$$\{\alpha_{mN}(x), V_{mN}(y), 0 \le x, y < \infty\} \stackrel{d}{=} \{\alpha_N^1(x), V_N^*(y), 0 \le x, y < \infty\}. \tag{89}$$

This completes the proof Theorem (3.2) with $n = mN$. Now, we prove the general case where $m(N-1) < n < mN$. It follows from (11) that

$$\begin{aligned}
&\{\alpha_n(x), 0 \le x < \infty\} \\
&\stackrel{d}{=} \quad \left\{N^{1/2}\left(G_{N,m}\left(x\frac{S_n}{mN}\right) - F^{(m)}(x)\right), 0 \le x < \infty\right\},
\end{aligned} \tag{90}$$

where

$$G_{N,m}(x) = \frac{1}{N}\sum_{i=1}^{N-1} \mathbb{1}_{\{Y_i < x\}} + \frac{1}{N}\mathbb{1}_{\{\sum_{\ell=(N-1)m+1}^n E_\ell < x\}}. \tag{91}$$

Moreover

$$\sup_{0\le x\le Q^{(m)}(a)} \left|G_{N,m}\left(x\frac{S_n}{mN}\right) - G_{N-1}\left(x\frac{S_n}{mN}\right)\right| \le \frac{1}{N} + \frac{1}{N(N-1)} \tag{92}$$

and

$$P\left(\left|\frac{S_n}{mN} - \frac{T_{N-1}}{m(N-1)}\right| > A_{18}N^{-1}(\log aN)\right) \le B_{18}N^{-\varepsilon}. \tag{93}$$

Taking

$$\mathcal{P} = P\left(\sup_{0\le x\le Q^{(m)}(a)} \left|N^{1/2}\left(G_{N,m}\left(x\frac{S_n}{mN}\right) - F^{(m)}(x)\right) - V_{N-1}^*(x)\right|\right. \tag{94}$$

$$\left. > A_{19}N^{-1/4}\left(\log aN\right)^{3/4}\right) \tag{95}$$

From (87) and (92) we have

$$\begin{aligned}
\mathcal{P} \quad \le \quad & P\left(\sup_{0\le x\le Q^{(m)}(a)} N^{1/2}\left|F^{(m)}\left(x\frac{S_n}{T_{N-1}}\right) - F^{(m)}(x)\right| > A_{20}N^{-1/2}(\log aN)\right) \\
&+P\left(\sup_{0\le x\le Q^{(m)}(a)} N^{1/2}\left|V_{N-1}^*\left(x\frac{S_n}{T_{N-1}}\right) - V_{N-1}^*(x)\right| > A_{21}N^{-1/2}(\log aN)\right) \\
&+B_3N^{-\varepsilon}.
\end{aligned}$$

As usual, by a first order the Taylor expansion we get

$$N^{1/2}\left|F^{(m)}\left(x\frac{S_n}{T_{N-1}}\right)-F^{(m)}(x)\right|=xf^{(m)}(x_N).N^{1/2}\left|\frac{S_n-T_{N-1}}{T_{N-1}}\right|, \tag{96}$$

where $|x_N-x|\le x\left|\frac{S_n-T_{N-1}}{T_{N-1}}\right|$. Lemma 4.2 and (93) now imply that

$$P\left(\left|\frac{S_n}{T_{N-1}}-1\right|>A_{22}N^{-1}(\log aN)\right)\le B_{22}N^{-\varepsilon}. \tag{97}$$

By arguing in a similar way as in the proof (69), we obtain that

$$P\left(\sup_{0\le x\le Q^{(m)}(a)}N^{1/2}\left|F^{(m)}\left(x\frac{S_n}{T_{N-1}}\right)-F^{(m)}(x)\right|>A_{20}N^{-1/2}(\log aN)\right)\le B_{20}N^{-\varepsilon}. \tag{98}$$

Now, by definitions (88), (98), and through a similar argument as that used at the end of the proof of Lemma 4.3, we get

$$P\left(\sup_{0\le x\le Q^{(m)}(a)}N^{1/2}\left|V^*_{N-1}\left(x\frac{S_n}{T_{N-1}}\right)-V^*_{N-1}(x)\right|>A_{21}N^{-1/2}(\log aN)\right)\le B_{21}N^{-\varepsilon}. \tag{99}$$

Then, by (98), (99) and (96) we have

$$P\left(\sup_{0\le x\le Q^{(m)}(a)}\left|N^{1/2}\left(G_{N,m}\left(x\frac{S_n}{mN}\right)-F^{(m)}(x)\right)-V^*_{N-1}(x)\right|\right. \tag{100}$$

$$\left.>A_{23}N^{-1/4}\left(\log aN\right)^{3/4}\right)\le B_{23}N^{-\varepsilon}. \tag{101}$$

Again, by Lemma 4.4.4 of (4) and (90), we can get a sequence of Gaussian processes $\{V_n(x);0\le x<\infty\},m(N-1)<n<mN,N=1,2\ldots$, such that for each N we have

$$\begin{aligned}&\{\alpha_n(x),V_n(y),0\le x,y<\infty\}\\ \stackrel{d}{=}\ &\left\{N^{1/2}\left(G_{N,m}\left(x\frac{S_n}{mN}\right)-F^{(m)}(x)\right),V^*_{N-1}(y),0\le x,y<\infty\right\}.\end{aligned}$$

This completes the proof of Theorem 3.2. ■

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